RELAXED r-COMPLETE PARTITIONS: AN ERROR-CORRECTING BACHET'S PROBLEM

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Abstract. Motivated by an error-correcting generalization of Bachet's weights problem, we define, classify and enumerate *relaxed r-complete partitions*. We show that these partitions enjoy a succinct description in terms of lattice points in polyhedra, with adjustments in the error being commensurate with translations in the defining hyperplanes. The enumeration of the minimal such partitions (those with fewest possible parts) can be achieved by the generating functions for $(r+1)$-*ary partitions*. This generalizes work of Park on classifying r-complete partitions and that of Rødseth on enumerating minimal r-complete partitions.

1. Introduction

First recorded by Fibonacci [7, On IIII Weights Weighing Forty Pounds] in 1202, Bachet's problem can be regarded as one of the earliest problems, if not the earliest, in the theory of partitions of integers. It asks: *what is the least number of pound weights that can be used on a scale pan to weigh any integral number of pounds from 1 to 40 inclusive, if the weights can be placed in either of the scale pans ?* Its solution consists of four parts and can be written as $40 = 1+3+9+27$ and is unique. Replacing 40 with any integer m, this problem has been generalized in a number of distinct ways: by MacMahon [3] in 1886; by Brown [1] in 1961 and by Park [6] in 1998. The latter was the first to describe all possible solutions to Bachet's problem as originally stated, when *40* is replaced with any integer m. A lively expository account of these various generalizations of Bachet's problem can be found in [5].

We will generalize Bachet's problem in a *relaxed* or *error-correcting* manner. We consider the following variant of it: *given a fixed integer weight of unknown weight l, weighing no more than 80 pounds, what is the least number of pound weights that can be used on a scale pan to discern l's value, if the weights can be placed in either of the scale pans ?* Here, we only need four parts and the partition $80 = 2+6+18+54$ will suffice. This is equivalent to saying that using the parts of $80 = 2+6+18+54$ we can weigh every integer between 1 and 80 on a two-scale pan, within an error of one. This leads us to the following definition.

Definition 1.1. A partition $m = \lambda_0+\lambda_1+\cdots+\lambda_n$ with the parts in increasing order is an *e-relaxed r-complete partition* (*(e, r)-partition* for short) if no $e+1$ consecutive integers between 0 and rm are absent from the set $\{\sum_{i=0}^{n} \alpha_i\lambda_i : \alpha_i \in \{0, 1, \ldots, r\}\}$. We call the partition *minimal* if n is as small as possible with this property.

Park [6], motivated by MacMahon's *perfect partitions* [3], called the $(0, r)$-partitions simply *r-complete partitions* and as a result of Park's work it can be shown that $40 = 1+3+9+27$ is the only minimal 2-complete partition of 40. This was also known to Hardy & Wright [2, §9.7]. To see

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the link between minimal 2-complete partitions and Bachet's problem we only need observe that for any 2-complete partition $m = \lambda_0 + \lambda_1 + \cdots + \lambda_n$, shifting the set $\{\sum_{i=0}^{n} \alpha_i \lambda_i : \alpha_i \in \{0,1,2\}\}$ by $-m$ we get the set $\{\sum_{i=0}^{n} \beta_i \lambda_i : \beta_i \in \{-1,0,1\}\}$ which is exactly the set of weights achievable by the parts of $m = \lambda_0 + \lambda_1 + \cdots + \lambda_n$ using the two-scale pan. Minimal 2-complete partitions are called *Bachet partitions* in [5].

Other simple observations include: if $\lambda_0 + \lambda_1 + \cdots + \lambda_n$ is a $(0,r)$-partition of m then $(e+1)\lambda_0 + (e+1)\lambda_1 + \cdots + (e+1)\lambda_n$ is an (e,r)-partition of $(e+1)m$. This partially explains the *doubling of parts* in solving our variant of Bachet's problem from the original. Similarly, every (e,r)-partition of m is both an $(e+1,r)$-partition and an $(e,r+1)$-partition of m. Finally, for every m and every (e,r), $m = \underbrace{1+1+\cdots+1+1}_{m \text{ times}}$ is an (e,r)-partition of m. We'll sometimes refer to the set $\{\sum_{i=0}^{n} \alpha_i \lambda_i : \alpha_i \in \{0,1,\ldots,r\}\}$ as the *r-cover* of $\lambda_0 + \lambda_1 + \cdots + \lambda_n$.

The r-complete partitions were classified and enumerated for every integer m by Park [6] and the minimal ones by Rødseth [8, 9]. This present piece will classify and enumerate the (e,r)-partitions for all e and r, carefully generalizing the analyses of both Park and Rødseth for the $e = 0$ case. We follow their arguments closely. Perhaps surprisingly, the error term e plays a relatively minor role when processed through both Park's and Rødseth's arguments, but significant attention to detail is required nonetheless. As one would expect, all of our results agree with those of Park and Rødseth when we set $e = 0$. Our first theorem is a classification of (e,r)-partitions.

Theorem 2.1 Let $m = \lambda_0 + \lambda_1 + \cdots + \lambda_n$ be a partition with $\lambda_0 \leq e+1$. Then $m = \lambda_0 + \lambda_1 + \cdots + \lambda_n$ is an (e,r)-partition if and only if $\text{ineq}_i : \lambda_i \leq (e+1) + r\sum_{j=0}^{i-1} \lambda_j$ holds for all $i \leq n$.

There is a nice polyhedral interpretation of the error term: the $(e+1,r)$-partitions of m can be attained from simultaneously shifting out the defining hyperplanes ineq_i for the (e,r)-partitions of m by one unit. Thus our choice of the adjective "relaxed" for our partitions: increasing the allowed error value e by 1 corresponds to a further relaxing of some of the defining hyperplanes of the polyhedron. The description also allows us to build up (e,r)-partitions sequentially. As for the minimal such partitions they can be described with the extra following condition.

Proposition 2.2 A minimal (e,r)-partition of m has exactly $\lfloor \log_{r+1}(\frac{rm}{e+1}) \rfloor + 1$ parts.

We next turn our attention to enumerating (e,r)-partitions. Defining $E_k(m)$ as the number of (e,r)-partitions of m with largest part of size k, we have that the number of (e,r)-partitions of m is be given by $\sum_{k=1}^{(rm+e+1)/(r+1)} E_k(m)$. We can enumerate (e,r)-partitions as follows.

Theorem 3.2 The number of (e,r)-partitions of m equals the coefficient of x^m in

$$\sum_{k=1}^{(rm+e+1)/(r+1)} \left[\frac{x^k}{\prod_{j=1}^{k}(1-x^j)} - x^{k-1} \sum_{i=3}^{k} \frac{x^{a_i}}{\prod_{j=i}^{k}(1-x^j)} E_{i-1}(i + a_i - 2) \right]$$

where $a_i := \left\lceil \frac{i-(e+1)}{r} \right\rceil$.

Observe that when $e \geq m$ then each $E_{i-1}(i + a_i - 2) = 0$ and $\sum_{k=1}^{(rm+e+1)/(r+1)} \frac{x^k}{\prod_{j=1}^{k}(1-x^j)}$ is essentially the generating function for $p(m)$, the number of partitions of m. This makes sense since

if $e \geq m$ then all partitions of m are (e,r)-partitions. Our original hope was that accounting for the (e,r)-partitions would somehow provide a back door to nice lower bounds on the partition function $p(m)$ but such desires do not readily appear attainable from our equation above, as it stands. We note too that Theorem 3.2 agrees with [6, Theorem 2.8] when $e = 0$.

Finally, we enumerate the minimal (e,r)-partitions. We begin by defining

$$F_n(x) = \sum_{j=0}^{\infty} f_n(j)x^j = \prod_{j=0}^{n} \frac{1}{1 - x^{(r+1)^j}} \qquad \text{with } F_{-1}(x) := 1$$

and

$$G_n(x) = \sum_{j=0}^{\infty} g_n(j)x^j = \sum_{j=0}^{n-1} \frac{x^{(r+1)^j - 1}}{1 - x^{2(r+1)^j}} F_j(x) F_{n-j-1}(x^{(2r+1)((r+1)^j)}) \qquad \text{with } G_0(x) := 0.$$

Theorem 4.4 For every $e \geq 0$ and $r \geq 2$, the number of minimal (e,r)-partitions of m equals

$$f_n(\frac{e+1}{r}((r+1)^{n+1} - 1) - m) - g_n(\frac{e+1}{r}((2r+1)(r+1)^{n-1} - 1) - 1 - m)$$

where $n := \lfloor \log_{r+1}(\frac{rm}{e+1}) \rfloor$ and $g(k) = 0$ when $k < 0$.

The restriction of minimality yields a more closed formula than in Theorem 3.2. This too agrees with [9, Theorem 2.1] when $e = 0$: there, for every n, both F_n and G_n are replaced with $F := \lim_{n\to\infty} F_n$ and $G := \lim_{n\to\infty} G_n$. This simplification to F and G also holds when $e \leq r$ but not so when $e > r$. Furthermore, it tells us that minimal (e,r)-partitions are still essentially counted by $(r+1)$-ary partitions, regardless of the error e. Geometrically, f_n is enumerating the number of points in $\mathbb{N}^{n+1}$ that satisfy the inequalities ineq_i of Theorem 2.1 and g_n is subtracting those points counted by f_n that do not satisfy the well ordering of parts needed for partitions. See [5] for an informal discussion of this, contrasting the $(0,2)$-partitions for $m = 16$ and $m = 25$.

Theorem 4.4 does not cover the case of $r = 1$. The minimal $(0,1)$-partitions were partially counted in [4] with the enumeration completed by Rødseth [8]. We finish with Theorem 4.5 which enumerates the minimal $(e,1)$-partitions, the formula for which is slightly more complicated than that of Theorem 4.4.

Specializing the above results for the $(1,2)$-partitions of 9 we see the following: using Theorem 2.1 and Proposition 2.2, of the thirty partitions of 9 all but seven – 9, $1+8$, $2+7$, $3+6$, $4+5$, $1+1+7$ and $3+3+3$ – are $(1,2)$-partitions. Of these 23 partitions, only five – $1+2+6$, $1+3+5$, $1+4+4$, $2+2+5$ and $2+3+4$ – are minimal. We justify these enumeration claims at the end of each relevant section.

2. Classifying the (e,r)-partitions

In this section we describe all possible (e,r)-partitions of a given integer m and the minimal such partitions amongst these. Much like the r-complete partitions they can be described in terms of lattice points in polyhedra. In fact, the defining hyperplanes for the polyhedra that cut out (e,r)-partitions arise from simply translating the defining hyperplanes (by a linear factor of e) that cut out the $(0,r)$-partitions. This can be seen from the following theorem.

Theorem 2.1. *Let $m = \lambda_0 + \lambda_1 + \cdots + \lambda_n$ be a partition with $\lambda_0 \leq e+1$. Then $m = \lambda_0 + \lambda_1 + \cdots + \lambda_n$ is an (e,r)-partition if and only if $\lambda_i \leq (e+1) + r\sum_{j=0}^{i-1} \lambda_j$, for all $i \leq n$.*

Proof. For sufficiency observe that if $\lambda_0 > (e+1)$ then $\{1,2,\ldots,e+1\}$ is a set of $e+1$ consecutive integers that are all absent from the r-cover of $\lambda_0+\lambda_1+\cdots+\lambda_n$. In much the same fashion, if for some $i \le n$, $\lambda_i > (e+1) + r\sum_{j=0}^{i-1}\lambda_j$ then the shifted set $r\sum_{j=0}^{i-1}\lambda_j + \{1,2,\ldots,e+1\}$ would also be omitted from the r-cover of the partition.

We show neccessity by inducting on the number of parts in the partition. If $n=0$ (the number of parts equals 1) then $\lambda_0 = m \le e+1$, in accordance with our hypothesis. Let $\lambda_0+\lambda_1+\cdots+\lambda_{n-1}$ be an (e,r)-partition onto which we append any part λ_n, not less than λ_{n-1} and with $\lambda_n \le (e+1)+r\sum_{j=0}^{n-1}\lambda_j$. We wish to show that every positive integer $l \le r(\lambda_0+\lambda_1+\cdots+\lambda_n)$ is within a distance of no greater than $e+1$ of some integer in the r-cover of $\lambda_0+\lambda_1+\cdots+\lambda_n$.

If $l \le r(\lambda_0+\lambda_1+\cdots+\lambda_{n-1})$ then by our inductive hypothesis we have nothing to show. So we can assume from here that we fix $l \le r\sum_{j=0}^{n}\lambda_j$ and $l > r\sum_{j=0}^{n-1}\lambda_j$. But in this case there will always exist an $1 \le \alpha_n \le r$ such that $(\alpha_n - 1)\lambda_n + r\sum_{j=0}^{n-1}\lambda_j < l \le \alpha_n\lambda_n + r\sum_{j=0}^{n-1}\lambda_j$ or $\alpha_n = \left\lceil \frac{l - r\sum_{j=0}^{n-1}\lambda_j}{\lambda_n} \right\rceil$. Again, since $l - \alpha_n\lambda_n \le r\sum_{j=0}^{n-1}\lambda_j$ our inductive hypothesis tells us that $l-\alpha_n\lambda_n$ is within distance $e+1$ of an integer in the r-cover of $\lambda_0+\lambda_1+\cdots+\lambda_{n-1}$ and so l must be within distance $e+1$ of an integer in the r-cover of $\lambda_0+\lambda_1+\cdots+\lambda_n$. $\square$

Next, we classify the *minimal* (e,r)-partitions. We've already observed that $\lambda_0 \le (e+1)$ and we can in turn note that $\lambda_1 \le (e+1) + r\lambda_0 \le (e+1)(r+1)$ and again in turn that $\lambda_2 \le (e+1)+r(\lambda_0+\lambda_1) \le (e+1)(r+1)^2$ and by an inductive argument it follows that

$$\lambda_i \le (e+1)(r+1)^i$$

for all (e,r)-partitions $\lambda_0+\lambda_1+\cdots+\lambda_n$.

Hence if $m = \lambda_0+\lambda_1+\cdots+\lambda_n$ is an (e,r)-partition then the sum of the parts in the partition cannot exceed $\sum_{i=0}^{n}(e+1)(r+1)^i = \frac{(e+1)}{r}((r+1)^{n+1}-1)$. That is,

$$m \le \frac{e+1}{r}((r+1)^{n+1}-1) < \frac{e+1}{r}(r+1)^{n+1} \ \text{ or } \ \log_{r+1}\left(\frac{rm}{e+1}\right) < n+1.$$

Since $n+1$ is an integer then the integer part of $\log_{r+1}(\frac{rm}{e+1})$ is strictly less than $n+1$ i.e. $\lfloor \log_{r+1}(\frac{rm}{e+1})\rfloor \le n$. This tells us that an (e,r)-partition of m must have *at least* $\lfloor \log_{r+1}(\frac{rm}{e+1})\rfloor + 1$ parts. This number will suffice.

Proposition 2.2. *A minimal (e,r)-partition of m has exactly $n+1 = \lfloor \log_{r+1}(\frac{rm}{e+1})\rfloor + 1$ parts.*

Proof. It suffices to show that for any integer m with $n := \lfloor \log_{r+1}(\frac{rm}{e+1})\rfloor$ the partition whose parts are exactly those in the multiset

$$\{\,(e+1),\,(e+1)(r+1),\,(e+1)(r+1)^2,\ldots,(e+1)(r+1)^{n-1},\,m - \frac{(e+1)}{r}((r+1)^n-1)\}$$

is an (e,r)-partition of m.

To see this first observe that $1+(r+1)+(r+1)^2+\cdots+(r+1)^{n-1}$ is a $(0,r)$-partition of $\frac{1}{r}((r+1)^n-1)$ and so $(e+1)+(e+1)(r+1)+(e+1)(r+1)^2+\cdots+(e+1)(r+1)^{n-1}$ is an (e,r)-partition of $\frac{(e+1)}{r}((r+1)^n-1)$. Next, for each $0 \le \alpha \le r$ the shifted set

$$\alpha\cdot(m - \frac{(e+1)}{r}((r+1)^n-1)) \;+\; \left\{\sum_{i=0}^{n-1}\alpha_i(e+1)(r+1)^i : \alpha_i \in \{0,1,\ldots,r\}\right\}$$

will not omit any consecutive $(e+1)$ integers and, since $m < \frac{e+1}{r}(r+1)^{n+1}$, the union over $1 \leq \alpha \leq r$ of these shifted sets not only range from 1 through to rm but do not omit any $(e+1)$ consecutive integers. This union, of course, is precisely the r-cover of the partition whose parts consist of the elements from the multiset. $\square$

We have already observed that if $\lambda_0 + \lambda_1 + \cdots + \lambda_n$ is a $(0,r)$-partition of m then $(e+1)\lambda_0 + (e+1)\lambda_1 + \cdots + (e+1)\lambda_n$ is a (e,r)-partition of $(e+1)m$. But the above proposition tells us that the *minimality* condition is also preserved by the multiplication of parts by $e+1$. This explains the *doubling of parts* in solving our variant of Bachet's problem with $80 = 2+6+18+54$ from the solution of $40 = 1+3+9+27$ to the original Bachet problem.

3. Counting (e,r)-partitions

This section is devoted to the construction of generating functions which will recursively yield the number of (e,r)-partitions for a given integer m. Suppose we have an (e,r)-partition of the integer $m-k$ given by $m-k = \lambda_0 + \lambda_1 + \cdots + \lambda_{l-1}$. By Theorem 2.1, this partition can be extended to an (e,r)-partition of m if and only if $\lambda_{l-1} \leq k \leq e+1+r(m-k)$. Letting $E_k(m)$ be the number of (e,r)-partitions with largest part of size k. Note that by the extension observation we readily attain $E_k(m) = \sum_{i=1}^{k} E_i(m-k)$ whenever $k \leq e+1+r(m-k)$.

Lemma 3.1. *If $0 \leq k \leq e+1+r(m-k)$ then $E_k(m) = E_k(m-k) + E_{k-1}(m-1)$.*

Proof. Let $k \leq e+1+r(m-k)$. It will suffice to show that $E_{k-1}(m-1) = \sum_{i=1}^{k-1} E_i(m-k)$. Since $k \leq e+1+r(m-k)$ then $k-1 \leq e+1+r[(m-1)-(k-1)]$ so, by Theorem 2.1, appending a part $k-1$ to any (e,r)-partition of $m-k$ of largest part bounded by $k-1$, yields a unique partition of $m-1$ with largest part exactly $k-1$. $\square$

By definition, we have $E_k(m) = 0$ when $k > m$ and we assume that $E_k(m) = 0$ when either $k < 0$ or $m < 0$. From the recursion $1 = E_1(1) = E_1(0) + E_0(0)$ we define $E_0(0) = 1$.

Note that the number of (e,r)-partitions of m is given by $\sum_{k=1}^{(e+1)+r(m-k)} E_k(m)$ and so accounting for each $E_k(m)$, as the next theorem does, suffices to enumerate (e,r)-partitions. It generalizes (and closely follows) Park's [6] enumeration of r-complete partitions.

Theorem 3.2. *For $\phi_k(x) := \sum_{m=1}^{\infty} E_k(m)x^m$ we have the generating function*

$$\phi_k(x) = \frac{x^k}{\prod_{j=1}^{k}(1-x^j)} - x^{k-1}\sum_{i=3}^{k} \frac{x^{a_i}}{\prod_{j=i}^{k}(1-x^j)} E_{i-1}(i+a_k-2).$$

where $a_i := \left\lceil \frac{i-(e+1)}{r} \right\rceil$.

Proof. Since for every m there is precisely one (e,r)-partition with largest part 1, we have $E_1(m) = 1$ for all $m \geq 0$. Hence $\phi_1(x) = \frac{x}{1-x}$. For fixed $k \geq 2$, $E_k(m) = 0$ unless m, by Theorem 2.1, is such that $k \leq (e+1)+r(m-k)$. In other words, unless $m \geq k+a_k$, were a_k is as defined above. Hence

$$\phi_k(x) = \sum_{m=k+a_k}^{\infty} E_k(m)x^m$$

Using Lemma 3.1 we can write $\phi_k(x)$ as the recursion

$$\begin{aligned}
\phi_k(x) \;=\; \sum_{m=k+a} E_k(m)x^m &= \sum_{m=k+a_k}^{\infty} E_k(m-k)x^m + \sum_{m=k+a_k}^{\infty} E_{k-1}(m-1)x^m \\
&= \sum_{m=a_k} E_k(m)x^{m+k} + \sum_{m=k+a_k-1} E_{k-1}(m)x^{m+1} \\
&= x^k \sum_{m=a_k} E_k(m)x^m + x \sum_{m=(k-1)+a_k} E_{k-1}(m)x^m
\end{aligned}$$

Since $E_k(m) = 0$ for $a \le m < a_k + k$ we can safely rewrite $\sum_{m=a} E_k(m)x^m$ as $\phi_k(x)$. On the other hand, $\sum_{m=(k-1)+a_k} E_{k-1}(m)x^m$ looks almost like $\phi_{k-1}(x)$, except that the sum starts from $k + a_k - 1$ instead of $k + a_{k-1} - 1$ as would be needed.

Note that, naively ignoring the ceiling function, we have $a_{k-1} + \frac{1}{r} = a_k$. Thus, with the ceiling function intact, $a_k - a_{k-1}$ equals either 0 or 1. When this difference equals 1 then

$$\phi_{k-1}(x) - \sum_{m=(k-1)+a_k} E_{k-1}(m)x^m = E_{k-1}(k + a_{k-1} - 1)x^{k+a_{k-1}-1} = E_{k-1}(k + a_k - 2)x^{k+a_k-2}.$$

And when $a_k - a_{k-1} = 0$ then $E_{k-1}(k+a_k-2) = 0$ since, by Theorem 2.1, there is no (e, r)-partition of $k+a_k-2 = k+a_{k-1}-2$ with largest part equal to $k-1$. So the above equation holds regardless of whether $a_k - a_{k-1}$ equals 0 or 1. Hence we have

$$\begin{aligned}
\phi_k(x) &= x^k\phi_k(x) + x\phi_{k-1}(x) - x^{k-1}x^{a_k}E_{k-1}(k + a_k - 2) \\
&= \frac{x}{1 - x^k}\phi_{k-1}(x) - x^{k-1}\frac{x^{a_k}}{1 - x^k}E_{k-1}(k + a_k - 2).
\end{aligned}$$

Repeating this recursion on $\phi_{k-1}(x)$ yields

$$\begin{aligned}
\phi_k(x) \;=\; &\frac{x^2}{(1 - x^k)(1 - x^{k-1})}\phi_{k-2}(x) \\
&- x^{k-1}\left[\frac{x^{a_{k-1}}}{(1 - x^{k-1})(1 - x^k)}E_{k-2}(k + a_{k-1} - 3) \;+\; \frac{x^{a_k}}{1 - x^k}E_{k-1}(k + a_k - 2)\right].
\end{aligned}$$

And repeating $k - 2$ more times, recalling that $\phi_0(x) = 1$ and $\phi_1(x) = \frac{x}{1-x}$ yields

$$\phi_k(x) = \frac{x^k}{\prod_{j=1}^{k}(1 - x^j)} - x^{k-1}\sum_{i=3}^{k}\frac{x^{a_i}}{\prod_{j=i}^{k}(1 - x^j)}E_{i-1}(i + a_k - 2).$$

Note that the sum begins at $i = 3$ since if $i = 2$ then the term $E_1(2 + a_2 - 2) = E_1(a_2)$ certainly arises in the recursion but $E_1(a_2) = 1$ if $\frac{2-(e+1)}{r} > 0$, or $1 > e + r$ which cannot occur since $e \ge 0$ and $r \ge 1$. Hence, $E_1(2 + a_2 - 2) = 0$ and need not be included in our expression. Similarly $E_0(a_1 - 1) = 0$ for all e and r. On the other hand, for $i = 3$, the term $E_2(1 + a_3) = 1$ precisely when $(e, r) = (0, 1)$, and zero otherwise. □

To account for the $(1, 2)$-partitions of 9 we begin with

$$\sum_{k=1}^{6}\frac{x^k}{\prod_{j=1}^{k}(1 - x^j)} \;=\; x + 2x^2 + 3x^3 + 5x^4 + 7x^5 + 11x^6 + 14x^7 + 20x^8 + 26x^9 + \cdots$$

Noting that $E_2(2) = 1$, $E_3(3) = 0$, $E_4(5) = 1$ and $E_5(6) = 0$ we have

$$\sum_{k=1}^{6} x^{k-1} \sum_{i=3}^{k} \frac{x^{a_i}}{\prod_{j=i}^{k}(1-x^j)} E_{i-1}(i + a_i - 2) \;=\; x^3 + x^4 + x^5 + 3x^6 + 2x^7 + 2x^8 + 3x^9 + \cdots$$

The coefficient of x^9 in their difference equals $26 - 3 = 23$, which agrees with the number of $(1, 2)$-partitions of 9 accounted for in the introduction.

4. Counting the minimal (e, r)-partitions

In the previous section we calculated the generating function for the (e, r)-partitions leading us naturally to ask the same for the minimal (e, r)-partitions. Equivalently, for a given fixed m we wish to count the number of (e, r)-partitions of m with exactly $n + 1 = \lfloor \log_{r+1}(\frac{rm}{e+1}) \rfloor + 1$ parts.

We will need to consider two separate cases, that of $r = 1$ and $r \geq 2$, but we need not immediately concern ourselves with these distinctions. In each case, Rødseth's arguments for the $e = 0$ case, [8] and [9] respectively, will be followed closely, including the use of much of Rødseth's terms. Similar to the previous section, the error term e acts as a witness in the sense that the procedure for counting minimal $(0, r)$-partitions does not differ greatly from that for counting minimal (e, r)-partitions.

Letting $q_n(m)$ be the number of (e, r)-partitions of m with $n + 1$ parts, we have that $q_n(m)$ equals the number of minimal (e, r)-partitions of m if $n = \lfloor \log_{r+1}(\frac{rm}{e+1}) \rfloor$. Thus, for each fixed n, we wish to study the generating function

$$Q_n(x) = \sum_{m=\frac{e+1}{r}((r+1)^n-1)+1}^{\frac{e+1}{r}((r+1)^{n+1}-1)} q_n(m)x^m = \sum_{|\lambda|} x^{|\lambda|}$$

where the latter sum is taken over all partitions $|\lambda| := \lambda_0 + \lambda_1 + \cdots + \lambda_n$ that satisfy $\lambda_i \leq (e + 1) + r(\lambda_0 + \lambda_1 + \cdots + \lambda_{i-1})$ for every $i = 0, 1, \ldots, n$.

Rather than computing $Q_n(x)$ directly, we will instead describe another collection of partitions whose enumeration will be equivalent to that of the minimal (e, r)-partitions. Setting

$$\mu_i := (e + 1)(r + 1)^i - \lambda_i$$

for each $i = 0, 1, \ldots, n$, our constraints on the λ_i's from Theorem 2.1 translate to

$$0 \leq \mu_0 \leq e \qquad \text{and} \qquad r\sum_{j=0}^{i-1} \mu_j \leq \mu_i \leq (e + 1)(r)(r + 1)^{i-1} + \mu_{i-1}.$$

We write $R_n(x)$ for

$$R_n(x) = \sum_{k\geq 0} r_n(k)x^k = \sum_{|\mu|} x^{|\mu|}$$

where the last sum is taken over all partitions μ that satisfy the translated μ constraints above. Recall that for two formal power series $\psi(x) := \sum_{i=0}^{\infty} \psi_i x^i$ and $\psi'(x) := \sum_{i=0}^{\infty} \psi'_i x^i$ we say that $\psi(x) = \psi'(x) + O(x^{N_0})$ if $\psi_i = \psi'_i$ for all $i < N_0$.

Proposition 4.1. *For all $n \geq 0$, $r_n(k) = q_n(\frac{e+1}{r}((r+1)^{n+1}-1)-k)$ whenever $k \leq (e+1)(r+1)^n-1$.*

Proof. Without including the terms of order $x^{(e+1)(r+1)^n}$ or higher we have

$$\begin{aligned} R_n(x) &= \sum_{k=0}^{(e+1)(r+1)^n-1} r_n(k)x^k = \sum_{|\mu|} x^{|\mu|} = x^{\frac{e+1}{r}((r+1)^{n+1}-1)} \sum_{|\lambda|} x^{-|\lambda|} \\ &= x^{\frac{e+1}{r}((r+1)^{n+1}-1)} Q_n(x^{-1}) = \sum_{m=\frac{e+1}{r}((r+1)^n-1)+1}^{\frac{e+1}{r}((r+1)^{n+1}-1)} q_n(m) x^{\frac{e+1}{r}(r+1)^{n+1}-1-m} \\ &= \sum_{k=0}^{(e+1)(r+1)^n-1} q_n(\frac{e+1}{r}((r+1)^{n+1}-1)-k)x^k. \end{aligned}$$

□

In view of Proposition 4.1 we focus our full attentions on understanding $R_n(x)$. We begin with the following two lemmas which, like Proposition 4.1, hold for all values of e and r.

Lemma 4.2. *$R_0(x) = \frac{1}{1-x} + O(x^{(e+1)})$ and $R_1(x) = \frac{1}{(1-x)(1-x^{r+1})} - \frac{x^{r(e+1)+1}}{(1-x)(1-x^2)} + O(x^{(e+1)(r+1)})$.*

Proof. For $n = 0$, since μ_0 can take any value between 0 and e then

$$R_0(x) = 1 + x + x^2 + \cdots + x^e = \frac{1-x^{e+1}}{1-x} = \frac{1}{1-x} + O(x^{(e+1)})$$

As for $n = 1$, $R_1(x) = \sum_{k=0}^{(e+1)(r+1)-1} r_1(k)x^k$ where

$$r_1(k) = \#\{\mu_0 + \mu_1 = k : 0 \le \mu \le e, r\mu_0 \le \mu_1 \le r(e+1) + \mu_0\}$$

and so

$$\begin{aligned} R_1(x) &= \sum_{\mu_0=0}^{e} \sum_{\mu_1=r\mu_0}^{r(e+1)-1} x^{\mu_0+\mu_1} = \sum_{\mu_0=0}^{e} x^{\mu_0} \frac{x^{r\mu_0}(1-x^{r(e+1)+\mu_0-r\mu_0+1})}{1-x} \\ &= \sum_{t=0}^{e} x^{t(r+1)} \frac{1-x^{r(e+1)+1+t(1-r)}}{1-x} = \frac{1}{1-x}\sum_{t=0}^{e}(x^{(r+1)})^t - \frac{x^{r(e+1)+1}}{1-x}\sum_{t=0}^{e} x^{t(r+1)+t(1-r)} \\ &= \frac{1}{(1-x)(1-x^{r+1})} - \frac{x^{r(e+1)+1}}{(1-x)(1-x^2)} + O(x^{(e+1)(r+1)}) \end{aligned}$$

□

Lemma 4.3. *For $n \ge 2$ we have*

$$R_n(x) = \frac{1}{1-x}R_{n-1}(x^{r+1}) - \frac{x^{(e+1)(r)(r+1)^{n-1}+1}}{(1-x)(1-x^2)}R_{n-2}(x^{2r+1}) + O(x^{(e+1)(r+1)^n})$$

Proof. Letting $n \ge 2$ we can write $R_n(x) = \sum_{\mu_0} \cdots \sum_{\mu_n} x^{\mu_0+\mu_1+\cdots+\mu_n}$ where the innermost sum simplifies to

$$\sum_{\mu_n=r\sum_{j=0}^{n-1}\mu_j}^{(e+1)(r)(r+1)^{i-1}+\mu_{n-1}} x^{\mu_0+\cdots+\mu_n} = x^{(r+1)(\mu_0+\cdots+\mu_{n-1})} \frac{1-x^{(e+1)(r)(r+1)^{n-1}+1-r(\mu_0+\cdots+\mu_{n-1})+\mu_{n-1}}}{1-x}$$

Hence we have the recursion

$$R_n(x) = \frac{1}{1-x}R_{n-1}(x^{r+1}) - \frac{x^{(e+1)(r)(r+1)^{n-1}+1}}{1-x}\sum_{\mu_0}\cdots\sum_{\mu_{n-1}} x^{\mu_0+\cdots+\mu_{n-2}+2\mu_{n-1}}$$

Repeating this process again on the innermost sum of the multisum yields

$$R_n(x) = \frac{1}{1-x}R_{n-1}(x^{r+1}) \;-\; \frac{x^{(e+1)(r)(r+1)^{n-1}+1}}{(1-x)(1-x^2)}R_{n-2}(x^{2r+1}) \;+\; O(x^{(e+1)(r+1)^n})$$

□

We can now prove the two main theorems of this section. With Proposition 4.1 in mind, the first enumerates (e, r)-partitions for every $e \geq 0$ and every $r \geq 2$.

4.1. **The $r \geq 2$ case.** Recalling our definitions of the generating functions $F_n(x)$ and $G_n(x)$, Proposition 4.1 tells us that the following theorem is equivalent to the statement given for it in the introduction.

Theorem 4.4. *For every $e \geq 0$ and $r \geq 2$, and for every $n \geq 1$ we have*

$$R_n(x) = F_n(x) - x^{r(e+1)(r+1)^{n-1}+1}G_n(x) + O(x^{(e+1)(r+1)^n}).$$

Proof. By Lemma 4.2 the result holds for $n = 1$. That is $R_1(x) = F_1(x) - x^{r(e+1)+1}G_1(x) + O(x^{(e+1)(r+1)})$. Let us, then, assume that $n \geq 2$ and that our claim holds for all cases strictly less than n. From Lemma 4.3,

$$R_n(x) = \underbrace{\frac{1}{1-x}R_{n-1}(x^{r+1})}_{(a)} - \underbrace{\frac{x^{(e+1)(r)(r+1)^{n-1}+1}}{(1-x)(1-x^2)}R_{n-2}(x^{2r+1})}_{(b)} + O(x^{(e+1)(r+1)^n}).$$

Using the inductive hypothesis on this recurrence relation, and using the functional equation

$$F_n(x) = \frac{1}{1-x}F_{n-1}(x^{r+1}),$$

term (a) simplifies to:

$$\begin{aligned}\frac{1}{1-x}R_{n-1}(x^{r+1}) &= \frac{1}{1-x}\left[F_{n-1}(x^{r+1}) - x^{(r+1)r(e+1)(r+1)^{n-2}+(r+1)}G_{n-1}(x^{r+1}) + O(x^{(r+1)(e+1)(r+1)^{n-1}})\right]\\ &= F_n(x) - x^{r(e+1)(r+1)^{n-1}+1}\left(\frac{x^r}{1-x}G_{n-1}(x^{r+1})\right) + O(x^{(e+1)(r+1)^n})\end{aligned}$$

In the same manner, term (b) $\frac{x^{r(e+1)(r+1)^{n-1}+1}}{(1-x)(1-x^2)}R_{n-2}(x^{2r+1})$ simplifies to

$$\begin{aligned}&\frac{x^{r(e+1)(r+1)^{n-1}+1}}{(1-x)(1-x^2)}\left[F_{n-2}(x^{2r+1}) - x^{(2r+1)r(e+1)(r+1)^{n-3}+(2r+1)}G_{n-2}(x^{2r+1}) + O(x^{(2r+1)(e+1)(r+1)^{n-2}})\right]\\ =&\frac{x^{r(e+1)(r+1)^{n-1}+1}}{(1-x)(1-x^2)}F_{n-2}(x^{2r+1}) - \frac{x^{(r(e+1)(r+1)^{n-1}+1)+(2r^2+r)(e+1)(r+1)^{n-3}+(2r+1)}}{(1-x)(1-x^2)}G_{n-2}(x^{2r+1}) + O(x^{(e+1)(r+1)^n})\\ =&\frac{x^{(e+1)(r)(r+1)^{n-1}+1}}{(1-x)(1-x^2)}F_{n-1}(x^{2r+1}) + O(x^{(e+1)(r+1)^n})\end{aligned}$$

The last equality (that of the term involving G_{n-2} being of order $O(x^{(e+1)(r+1)^n})$) arises as follows: since $r \geq 2$ then $2r^2 + r \geq (r+1)^2$ and so

$$r(e+1)(r+1)^{n-1} + (2r^2+r)(e+1)(r+1)^{n-3} \geq r(e+1)(r+1)^{n-1} + (e+1)(r+1)^{n-1} = (e+1)(r+1)^n$$

Combining parts (a) and (b), and using the functional equation

$$G_n(x) = \frac{x^r}{1-x}G_{n-1}(x^{r+1}) + \frac{1}{(1-x)(1-x^2)}F_{n-1}(x^{2r+1}),$$

we obtain the relation $R_n(x) = F_n(x) - x^{r(e+1)(r+1)^{n-1}+1}G_n(x) + O(x^{(e+1)(r+1)^n})$ as claimed. $\square$

We verify that this formula fits our claim of there being five minimal $(1,2)$-partitions of 9. First we have $F_2(x) = 1 + x + x^2 + \cdots + 7x^{14} + 9x^{15} + 9x^{16} + 9x^{17} + 12x^{18} + \cdots$. Next, $G_2(x) = 1 + x + 3x^2 + 3x^3 + 4x^4 + 6x^5 + 7x^6 + \cdots$. Using Theorem 4.4, we have $q_2(9) = r_2(26-9) = r_2(17)$ which implies that the number of minimal $(1,2)$-partitions of 9 equals the coefficient of x^{17} in

$$\begin{aligned} R_2(x) = F_2(x) - x^{13}G_2(x) &= (1 + x + x^2 + \cdots + 7x^{14} + 9x^{15} + 9x^{16} + 9x^{17} + 12x^{18} + \cdots) \\ &\quad - (x^{13} + x^{14} + 3x^{15} + 3x^{16} + 4x^{17} + 6x^{18} + 7x^{19} + \cdots) \end{aligned}$$

which equals $9 - 4 = 5$ as claimed.

4.2. **The $r = 1$ case.** Note that in the proof of Theorem 4.4, the claim of the term involving G_{n-2} being of order $O(x^{(e+1)(r+1)^n}))$ arose from $r \geq 2$ which implied that $2r^2+r \geq (r+1)^2$, which does not hold for $r = 1$. We thus need to make a distinct argument for counting the minimal (e,r)-partitions for $r = 1$. To do so, we define the following two generating functions, with $F_n(x) = \prod_{i=0}^n \frac{1}{1-x^{2^i}}$ as before:

$$D_n(x) = \sum_{j=0}^{\infty} d_n(j)x^j = \sum_{j=0}^{n-1} x^{2^j-1}F_{j+1}(x)F_{n-j-2}(x^{3\cdot 2^j}) \qquad \text{with } D_0(x) := 0$$

and

$$D_n^*(x) = \sum_{j=0}^{\infty} d_n^*(j)x^j = \sum_{j=0}^{n-1} x^{2^{j+2}-4}F_{j+1}(x)D_{n-j}(x^{3\cdot 2^j}) \qquad \text{with } D_0^*(x) := 0.$$

Theorem 4.5. *For $r = 1$ and for every $n \geq 3$ we have the following generating function for the minimal $(e,1)$-partitions:*

$$R_n(x) = F_n(x) - x^{2^{n-1}(e+1)+1}D_n(x) + x^{7\cdot 2^{n-3}(e+1)+4}D_{n-2}^*(x) + O(x^{2^n(e+1)}).$$

Proof. We can rewrite Lemma 4.2 for $r = 1$ as $R_0(x) = F_0(x)$ and $R_1(x) = F_1(x) - x^{(e+1)+1}F_1(x)$. All these terms, and all that follow, are correct up to $O(x^{2^n(e+1)})$. Using the recursion of Lemma 4.3, and the functional equation $F_n(x) = \frac{1}{1-x}F_{n-1}(x^2)$ we get $R_2(x) = F_2(x) - x^{2(e+1)+1}[xF_2(x) + F_1(x)F_0(x^3)]$ and

$$\begin{aligned} R_3(x) &= F_3(x) - x^{2^2(e+1)+1}[x^3F_3(x) + xF_2(x)F_0(x^6) + F_1(x)F_1(x^3)] + x^{7(e+1)+4}[F_1(x)F_1(x^3)] \\ &= F_3(x) - x^{2^2(e+1)+1}D_3(x) + x^{7(e+1)+4}D_1^*(x) \end{aligned}$$

This confirms our claim for $n = 3$. We now proceed by induction for all $n \geq 4$. From Lemma 4.3,

$$R_n(x) = \frac{1}{1-x}R_{n-1}(x^2) \; - \; x^{2^{n-1}(e+1)+1}F_1(x)R_{n-2}(x^3) \; + \; O(x^{2^n(e+1)}).$$

Using the inductive hypothesis on R_{n-1} and R_{n-2}, and the functional equation for F_n, we have (once again, up to $O(x^{2^n(e+1)})$)

$$R_n(x) = \frac{1}{1-x}\left[F_{n-1}(x^2) - x^{2^{n-1}(e+1)+1}xD_{n-1}(x^2) + x^{7\cdot 2^{n-3}(e+1)+4}x^4D^*_{n-3}(x^2)\right]$$

$$- x^{2^{n-1}(e+1)+1}\cdot \quad F_1(x)\left[F_{n-2}(x^3) - x^{3\cdot 2^{n-3}(e+1)+3}D_{n-2}(x^3) + x^{7\cdot 3\cdot 2^{n-5}(e+1)+12}D^*_{n-4}(x^3)\right]$$

and so

$$R_n(x) = F_n(x) - x^{2^{n-1}(e+1)+1}\frac{x}{1-x}D_{n-1}(x^2) \quad + x^{7\cdot 2^{n-3}(e+1)+4}\frac{x^4}{1-x}D^*_{n-3}(x^2)$$

$$- x^{2^{n-1}(e+1)+1}F_1(x)F_{n-2}(x^3) \quad + x^{7\cdot 2^{n-3}(e+1)+4}F_1(x)D_{n-2}(x^3) - \quad O(x^{2^n(e+1)}).$$

The following functional equations can be easily verified

$$D_n(x) = \frac{x}{1-x}D_{n-1}(x^2) + F_1(x)F_{n-2}(x^3) \qquad \text{and} \qquad D^*_n(x) = \frac{x^4}{1-x}D^*_{n-1}(x^2) + F_1(x)D_n(x^3),$$

and when combined with our expression for $R_n(x)$ we get $R_n(x) = F_n(x) - x^{2^{n-1}(e+1)+1}D_n(x) + x^{7\cdot 2^{n-3}(e+1)+4}D^*_{n-2}(x) + O(x^{2^n(e+1)})$. □

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